A STUDY OF SERIES AND APPROXIMATION OF FUNCTIONS BY SUMMABILITY TECHNIQUES

RESEARCH PROPOSAL
&
PROPOSED PLAN

CONCERNING PH.D. DEGREE IN MATHEMATICS

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1. INTRODUCTION:

Generally different types of series occur frequently in many problems of mathematics and of real world. Infinite series are useful in all types of problems where the need of studying their convergence or divergence is very important. If a series used in an investigation is not convergent it may lead to absurd result. Hence it is essential to discuss about infinite series in the beginning.

A collection of numbers $u_1 + u_2 + ... + u_n + ...$ is called a sequence if for every positive integer n there exist a number u_n .

A series $u_1 + u_2 + ... + u_n + ... = \sum_{k=1}^{n} u_n$ is said to be a finite series as the number of terms is limited and if number of terms in a series is infinitely large then it is said to be infinite series, which is defined as

$$u_1 + u_2 + ... + u_n + ... = \sum_{n=1}^{\infty} u_n$$

If we take the sum of first n terms of the series $\sum_{n=1}^{\infty} u_n$, we call the sum as n^{th} partial sum and write $S_n = u_1 + u_2 + ... + u_n = \sum_{k=1}^{n} u_k$.

The series $\sum_{n=1}^{\infty} u_n$ is said to be convergent if $\lim_{n\to\infty} S_n$ is a finite number, say S. Then S is called the sum of the series $\sum_{n=1}^{\infty} u_n$.

If $\lim_{n\to\infty} S_n = +\infty$ *or* $-\infty$ then $\sum_{n=1}^{\infty} u_n$ is said to be divergent.

If $\lim_{n\to\infty} S_n$ does not approach to a unique limit then $\sum_{n=1}^{\infty} u_n$ is said to be oscillatory.

The sequences

$$1, (2!)^{\frac{1}{2}}, (3!)^{\frac{1}{3}}, (4!)^{\frac{1}{4}} \ldots$$

$$-1, -2, -3, -4 \ldots$$

$$1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots, \frac{1}{\sqrt{n}}, \ldots$$

$$\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \ldots, \frac{n}{n^2 - 1}, \ldots$$

$$\frac{1}{\log_e 2}, \frac{1}{\log_e 3}, \frac{1}{\log_e 4}, \ldots$$

$$\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}}, \ldots$$

are divergent sequences and the following sequences

$$1 - 2 + 3 - 4 + \ldots$$

$$1 - 2^n + 3^n - 4^n + \ldots$$

are oscillatory.

The series

$$1 + e^{i\theta} + e^{2i\theta} + \ldots$$

$$\frac{1}{2} + \cos\theta + \cos 2\theta + \ldots$$

are divergent for all θ and

$$\sin\theta + \sin 2\theta + \sin 3\theta + \ldots$$

is divergent except, when θ is a multiple of π, when it converges to the sum 0. Convergence denotes the approach towards definite value. It describes the limiting behavior, particularly to an infinite sequence or series, towards some limit. To assert convergence is to claim the existence of such a limit, which may be itself unknown. For a fixed standard of accuracy, however we can always be sure to be within that limit, provided we have gone far enough.

Divergence may be some kind of oscillation, unrestricted growth (recognized in the case of an infinite limit) or chaotic behavior. An infinite series that is divergent cannot be used for meaningful computations of its value. An infinite sequence of partial sum of a divergent series does not has a limit. It is quite true that the attention paid to divergent

series steadily diminished during the first eighty years of the nineteenth century, and that it had almost reached the vanishing point during the last decade of that period. This was due, however, to a variety of factors of which the Abel-Cauchy dictum was only one element. The gradual development of one present day notions of rigor was perhaps the dominant influence. The general standards of rigor set forth by Abel and Cauchy played a very important role in this development. Cauchy's monumental work 'Course d' Analyse Algebrique in 1821 and Abel's [1] work on binomial series in 1826 and thus their indirect influence on the study of divergent series was in itself of considerable importance.

The extent to which the study of divergent series was continued, both in England and Germany, during the first sixty years of the nineteenth century, has been clearly pointed out by Burkhardt [5] in his paper in 1911.

Since there was a steady development of notions of rigor during the same period, one may naturally inquire why the rigorous treatment of divergent series did not get an early start.

The reason was not far to seek. From the point of view of relative frequency and relative simplicity, convergent series bear somewhat the same relationship to series in general that analytic functions do to general functions. Just as it was not feasible for mathematicians to undertake the study of very general types of functions until they had a considerable understanding of the highly special but particularly important class of analytic functions, so it was exceedingly difficult to build a rigorous theory of divergent series before the theory of convergent series had reached some degree of completeness.

The attainment of this simpler objective was a task of no mean difficulty and it absorbed the main energy of students of series during the greater part of the nineteenth century. It is an interesting co-incidence that Pringsheim's [25] comprehensive study of convergent series with positive term was published in the same year 1890 as Cesaro's fundamental paper on the multiplication of series [6]. The latter paper, as we all know, was one of the prime successes of the work of the present century in the field of divergent series.

After persistent efforts attempted by several mathematicians, it was in closing decades of the 19^{th} century and in the early ones of the 20^{th} century that the satisfactory methods were devised through processes closely connected with Cauchy's concept of convergence, certain values which may be called their sums in a less direct way. Such

processes of summation of series, which were formerly, taboos being divergent have now resulted in the modern rigorous theory of summability. Summability itself has become an important branch of mathematics.

If a series $u_1 + u_2 + u_3 .....$ is not convergent i.e. if $S_n = u_1 + u_2 + u_3 ..... + u_n$ does not tend to a limit, it is sometimes possible to associate with series sum in a less direct way. The simplest such method is "summability by arithmetic means." We take the arithmetic mean

$$\sigma_n = \frac{s_1 + s_2 + ... + s_n}{n}$$

of the partial sums of the given series. If $S_n \to S$, then also $\sigma_n \to S$ for if $S_n = S + \delta_n$ then

$$\sigma_n = S + \frac{\delta_1 + \delta_2 + ... + \delta_n}{n}$$

and the last term tends to zero if $\delta_n \to 0$. But σ_n may tend to a limit even though S_n does not. For example consider the series

$$1 - 1 + 1 - 1 + ...$$

here the partial sums S_n are alternately 1 and 0 and it is easily seen that $\sigma_n \to \frac{1}{2}$.

A series for which σ_n tends to a limit is said to be summable by arithmetic means, or by Cesaro's means of the first order or (C, 1).

The concept of convergence having been generalized, it was but natural to enquire if the notion of absolute convergence was capable of similar generalization. The answer to this has been found in affirmative and in fact, just as the notion of convergence was instrumental to the development of the various summability methods, so also, by analogy, the concept of absolute convergence led to the formulation of the processes of absolute summability. The method of absolute summability was first introduced by Woronoi [38] but it more closely identified with the name of Nörlund [19]. The absolute summability of a Fourier series is discussed by several researchers like Fekete [8], Prasad [23, 24], Hyslop [10] etc.

Lorentz's [17] for the first time in 1948, defined that a bounded sequence $\{S_n\}$ is said to almost convergent to a limit S if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=m}^{n+m} S_\nu = S$$

uniformly with respect to m, where S_ν is the partial sum of the series $\sum_{n=1}^{\infty} u_n$. It is easy to see that convergent sequence is almost convergent and the limits are the same. The idea of almost convergent led to the formulation of almost summability methods.

During sixth decade of twentieth century Saxena [33] and Rajagopal [29] established some interesting results on uniform, harmonic and Nörlund summability of Fourier series.

Nörlund summability (N, p_n) of Fourier series and conjugate series of Fourier series has been discussed by number of researchers like Pati [21], Tripathi [36] and Rajagopal [29] etc.

Generalised Nörlund summability (N, p, q) of Fourier series and its conjugate series was discussed by Khare [13] in 1990. (N, p, q) summability reduces to (N, p_n) summability for $q_n = 1 \ \forall\, n$ *and to Riesz* $(\overline{N}, q_n)$ summability for $p_n = 1 \quad \forall \quad n.$

K^λ - summability was first introduced by Karamata [11]. Thereafter Lotosky [18] considered a particular case $\lambda = 1$ for this summability. Later on Agnew [2] has done an intensive study of similar methods. *Vuĉkoviĉ* [37] applied K^λ method for summability of Fourier series. After his result, a number of researchers like Kathal [12], Ojha [20], Tripathi & lal [35], Lal& Pratap [15] have produced their work on K^λ summability .
After quite a good amount of work done using single summability methods different product summability methods were put on sound foundation till the beginning of 21st century. In present work we shall try to study different infinite series by different summability methods.

Summability theory can also be applied in other branches of mathematics like approximation theory, functional analysis etc., so in present work we shall study the

degree of approximation of different functions by product summability means of its series.

The applicability of summability techniques can also be illustrated by Fejer theorem in Fourier analysis and Borel theorem of large numbers in probability theory.

2. PURPOSE OF STUDY:

In this work we shall study derived series of a Fourier series by K^{λ} - summability method. The concept of K^{λ} - summability was first introduced by Karamata [11] and reintroduced for a special case $\lambda=1$ by Lotosky [18]. After the work of Agnew [2] an intensive study of similar methods has taken place. K^{λ} - summability of Fourier series was first discussed by *Vučkovič* [37]. Thereafter a number of researchers like Kathal [12], Ojha [20], Tripathi & lal [35], Lal& Pratap [15] have generalized the result of *Vučkovič* [37]. Lal [14], generalizing the result of Kathal [12] has studied conjugate series of a Fourier series by K^{λ} – summability method. The purpose of our work is to establish a theorem on K^{λ} – summability of derived series of a Fourier series.

Till the beginning of the 21st century, the concept of product summability methods were put on sound foundation and a few work has been done so far. The purpose of our work is to study different infinite series by product summability methods. A good amount of work has been done to study a Fourier series using (N, p_n) and (N, p, q) summability respectively by several researchers like Singh [34], Rajagopal [29], Saxena [31, 32], Lal [14], Pandey [22], Khare [13].

Lal [14] discussed (C, 1)(E,1) summability of a Fourier series. In this work we shall prove a theorem on $(N, p, q)(E,1)$ summability of a Fourier series. $(N, p, q)(E,1)$ summability includes $(N, p_n)(E,1)$ and (C, 1)(E, 1) summabilities as particular cases.

The objective of this work is also to study a Fourier series by (C, 1)(E, 1) summability method so that a new theorem on (C, 1)(E, 1) summability of a Fourier series will be proved, which includes result of Lal [14].

In this work we shall also prove quite a new theorem on (E, 1)(C, 1) summability of a Fourier series as no work has been found in this direction. In this work concepts of summability theory will be applied in other branches of mathematics like Approximation

Theory and Functional Analysis for functions $f \in Lip\alpha$. The degree of approximation by Cesaro mean and by Norlund mean of Fourier series of f have been studied by number of researchers like Alexits [3], Sahney and Goel [30], Chandra[7], Qureshi [26, 27] and Qureshi and Neha [28].

In this work we shall study the degree of approximation of $Lip\alpha$ class by (C, 1)(N, p_n) means of conjugate series of its Fourier series. We shall also prove a result on degree of approximation of the Lip-schitz functions by (C, 1)(N, p_n) mean of derived series of a Fourier series. Therefore several previous results will be the particular cases of our proposed problems.

3. PROPOSED WORK:

We propose to investigate on the following problems:

1. On K^{λ} - summability of derived series of a Fourier series.
2. $(N, p_n)(E,1)$ summability of Fourier series.
3. (C,1)(E,1) summability of a Fourier series.
4. (E,1)(C,1) summability of a Fourier series.
5. On the degree of approximation of a function belonging to $Lip\alpha$ class by (C,1)(N, p_n) summability of conjugate series of a Fourier series.
6. On the degree of approximation of a function belonging to $Lip\alpha$ class by (C,1)(N, p_n) summability of derived series of a Fourier series.

4. RESOURCES OF LITERATURE:

Literature is being studied from following books:

1. Divergent series by G. H. Hardy.
2. Trigonometrical series by Antony Zygmund .
3. Real Analysis by H. L. Royden.
4. Real Analysis by Saran, Chadda and Gupta.
5. Problems in mathematical analysis by B. Demidovich.

We are also consulting the research papers published in the following journals:

1. Arab journal of mathematics and mathematical sciences

2. Journal of inequality and applications
3. Canadian mathematical society
4. Science direct
5. Quarterly journal of mathematics
6. Bulletin of Calcutta mathematical society
7. Tamkang journal of mathematics
8. Proceeding American mathematical society
9. Indian journal of pure and applied mathematics
10. Journal of Indian mathematical society
11. The Australian journal of mathematical analysis and applications
12. Matematicheskii sbornik
13. Scopus
14. Informa world
15. Iop electronic journals

5. METHODOLOGY:

FOURIER SERIES:

Let $f(t)$ be the 2π-periodic and Lebesgue integrable function of t over the interval $(-\pi,\pi)$. Then

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(t) \qquad (1)$$

is called Fourier series, here the coefficients a_0, a_n *and* b_n are called Fourier coefficients and given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx \text{ , here, n = 1, 2, 3...}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx \text{ , here, n = 1, 2, 3...}$$

CONJUGATE FOURIER SERIES:

Conjugate series of a Fourier series (1) is given by

$$\sum_{n=1}^{\infty}(a_n \sin nt - b_n \cos nt) = \sum_{n=0}^{\infty} B_n(t) \tag{2}$$

DERIVED FOURIER SERIES:

The series

$$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) = -\sum_{n=1}^{\infty} nB_n(t) \tag{3}$$

which is obtained by differentiating (1) term-by-term and is called the first derived series of Fourier series of $f(t)$.

DERIVED CONJUGATE FOURIER SERIES:

The series

$$\sum_{n=1}^{\infty} n(a_n \cos nt + b_n \sin nt) \tag{4}$$

which is obtained by differentiating (2) term-by-term and is called derived series of conjugate Fourier series.

(C, 1) SUMMABILITY:

Let $\sum_{n=0}^{\infty} u_n$ is a series. Then $\sum_{n=0}^{\infty} u_n$ is said to be summable (C,1) to the sum S if

$$\lim_{n\to\infty} \frac{S_0 + S_1 + \ldots\ldots S_n}{n+1} \to S .$$

(E, 1) SUMMABILITY:

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series whose nth partial sum is given by S_n.

If $$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} S_k \to S, \quad as \quad n \to \infty$$

then the series $\sum_{n=0}^{\infty} u_n$ is said to summable to S by (E,1) summability and E_n^1 is called the (E,1) transform of S_n.

NÖRLUND SUMMABILITY:

Let $p_m \geq 0$ be a sequence where, $P_n = p_0 + p_1 + ... + p_n$.

Now we define t_m by

$$t_m = \frac{p_m s_0 + p_{m-1} s_1 + ... + p_0 s_m}{p_0 + p_1 + ... + p_n}$$

If $t_m \to \infty$ when $m \to \infty$ and $s_n = a_0 + a_1 + ... + a_n$,

then we write $S_n \to S, \quad \sum a_n = S(N, p_n)$

REISZ SUMMABILITY:

The transformation given by

$$t_m = \frac{p_1 s_1 + p_2 s_2 + ... + p_m s_m}{p_0 + p_1 + ... + p_m}$$

If $t_m \to \infty$ when $m \to \infty$ and $s_n = a_0 + a_1 + ... + a_n$,

then we write $S_n \to S, \quad \sum a_n = S(\overline{N}, q_n)$.

GENERALISED NÖRLUND SUMMABILITY:

Let $\{p_n\}$ and $\{q_n\}$ be the sequences of constants, real or complex, such that

$$P_n = p_0 + p_1 + p_2 + ... + p_n = \sum_{\nu=0}^{n} p_\nu \to \infty, \ as \ n \to \infty$$

$$Q_n = q_0 + q_1 + q_2 + ... + q_n = \sum_{\nu=0}^{n} q_\nu \to \infty, \ as \ n \to \infty$$

$$R_n = p_0 q_n + p_1 q_{n-1} + ... + p_n q_0 = \sum_{\nu=0}^{n} p_\nu q_{n-\nu} \to \infty, \ as \ n \to \infty$$

Now convolution of two sequences $\{p_n\}$ and $\{q_n\}$ is defined as

$$R_n = (p * q)_n = \sum_{k=0}^{n} p_{n-k} q_k .$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with the sequence of its nth partial sums $\{s_n\}$.

We write $$t_n^{p,q} = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k s_k$$

If $R_n \neq 0$, for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$.

If $t_n^{p,q} \to S, \ as \ n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or sequence $\{s_n\}$ is summable to S by generalized Nörlund method (Borwein [4]) and is denoted by

$$S_n \to \mathrm{S}\ (N,p,q)$$

Particular cases of (N,p,q) mean are:

(i) (N,p,q) mean reduces to (N,p_n) summability mean if $q_n = 1, \forall\, n$.

(ii) (N,p,q) method reduces to $(\overline{N},q_n)$ if $p_n = 1, \forall\, n$.

$(N,p_n)(C,1)$ ***SUMMABILITY:***

Let $\sum_{n=0}^{\infty} u_n$ ***be an infinite series whose*** n^{th} ***partial sum is given by*** S_n. ***We write***

$$S_n = \frac{s_1 + s_2 + \ldots\ldots + s_n}{n+1}$$

Let $\{p_n\}$ ***be the sequence of real constants such that***

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty, \ as \ n \to \infty$$

If $$(NC)_n^1 = \frac{1}{p_n}\sum_{\nu=0}^{n} p_\nu \delta_{n-\nu} \to S, \quad as \quad n \to \infty$$

then $\sum_{n=0}^{\infty} u_n$ ***or sequence*** $\{S_n\}$ ***is said to be*** $(N,p_n)(C,1)$ ***summable to S.***

$(N,p_n)(E,1)$ **SUMMABILITY:**

Let $$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k$$

If $E_n^1 \to s, as\ n \to \infty$ then the series $\sum_{n=0}^{\infty} u_n$ with partial sums S_n is said to be (E,1) summable to S(Hardy [9]).

Let $\{p_n\}$ be the sequence of real constants such that $p_0 > 0,\ p_n \geq 0,\ n \geq 1$

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty,\ as\ n \to \infty$$

Then (N, p_n) transform (Hardy, 1949) of the (E, 1) transform E_n^1 defines the $(N, p_n)(E,1)$ transform of the partial sum S_n of series $\sum_{n=0}^{\infty} u_n$.

If $$(NE)_n^1 = \frac{1}{p_n}\sum_{k=0}^{n} p_k E_k^1$$

$$= \frac{1}{p_n}\sum_{k=0}^{n} p_k \frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu} S_\nu \to S,\ as\ n \to \infty$$

Where E_n^1 denotes the (E, 1) transform of S_n then $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(N, p_n)(E,1)$ mean.

$(E,1)(N, p_n)$ **SUMMABILITY:**

If $E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k \to S,\ as\ n \to \infty$ then the infinite series $\sum_{n=0}^{\infty} u_n$ with partial sums S_n is said to summable (E, 1) to the definite number S.

Let $\{p_n\}$ be the sequence of real constants such that $p_0 > 0,\ p_n \geq 0,\ n \geq 1$

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty,\ as\ n \to \infty$$

now we define

$$t_m = \frac{p_m s_0 + p_{m-1} s_1 + \ldots\ldots + p_0 s_m}{p_0 + p_1 + \ldots\ldots + p_n}$$

If $t_m \to \infty$ when $m \to \infty$ and $s_n = u_0 + u_1 + \ldots\ldots + u_n$,

then we write $S_n \to S,\ \sum u_n = S(N, p_n).$

The (E, 1) transform of the (N, p_n) transform defines the $(E,1)(N, p_n)$ transform of the partial sum S_n of series $\sum u_n$.

If $$(E\,N)_n^1 = \frac{1}{2^n}\sum_{k=0}^{n} {}^nC_k \frac{1}{p_k}\sum_{\nu=0}^{k} p_{k-\nu}s_k$$

then $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(E,1)(N,p_n)$ summability method.

$(N,p,q)(E,1)$ SUMMABILITY:

Let $\{p_n\}$ and $\{q_n\}$ be the sequences of constants, real or complex, such that

$$P_n = p_0 + p_1 + p_2 + \ldots + p_n = \sum_{k=0}^{n} p_k \to \infty, \; as \; n \to \infty$$

$$Q_n = q_0 + q_1 + q_2 + \ldots + q_n = \sum_{k=0}^{n} q_k \to \infty, \; as \; n \to \infty$$

$$R_n = p_0 q_n + p_1 q_{n-1} + \ldots + p_n q_0 = \sum_{k=0}^{n} p_k q_{n-k} \to \infty, \; as \; n \to \infty$$

Now convolution of two sequences $\{p_n\}$ and $\{q_n\}$ is defined as

$$R_n = (p * q)_n = \sum_{k=0}^{n} p_{n-k} q_k \, .$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with the sequence of its nth partial sums $\{s_n\}$.

We write,

$$t_n^{p,q} = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k s_k$$

If $R_n \neq 0$, for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$.

If $t_n^{p,q} \to S, \; as \; n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or sequence $\{s_n\}$ is summable to S by generalized Nörlund method (Borwein [4]) and is denoted by

$$S_n \to \mathrm{S}\,(N,p,q).$$

Now Let $$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n} \binom{n}{k} s_k$$

If $E_n^1 \to s, as\ n \to \infty$

then the series $\sum_{n=0}^{\infty} u_n$ is said to be (E,1) summable to s (Hardy [8])

$$t_n^{p,q,E} = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k E_k^1$$

$$= \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k \cdot \frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu} s_\nu$$

If $t_n^{p,q,E} \to \infty, as\ n \to \infty$ then the series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{S_n\}$ is summable to S by (N,p,q)(E,1) summability method.

$(C,1)(E,1)$ SUMMABILITY:

The $(C,1)$ transform of the $(E,1)$ transform E_n^1 defines the $(C,1)(E,1)$ transform of the partial sum S_n of series $\sum_{n=0}^{\infty} u_n$.

Let $\sum_{n=0}^{\infty} u_n$ is a series. Then $\sum_{n=0}^{\infty} u_n$ is said to be summable (C,1) to the sum S if

$$\lim_{n\to\infty} \frac{S_0 + S_1 + \ldots\ldots S_n}{n+1} \to S$$

Thus if $(CE)_n^1 = \frac{1}{n}\sum_{k=0}^{n} E_k^1$

$$= \frac{1}{n}\sum_{k=0}^{n} \frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r} S_r \to S \quad , \quad as \quad n \to \infty$$

where E_n^1 denotes the $(E,1)$ transform of S_n i.e.

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k$$

and S is a finite constant then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(C,1)(E,1)$ means i.e.

$(CE)_n^1 \to S$, $(C,1)(E,1)$ $\quad as \quad n \to \infty$.

$(E,1)(C,1)$ SUMMABILITY:

The $(E,1)$ transform of the $(C,1)$ transform E_n^1 defines the $(E,1)(C,1)$ transform of the partial sum S_n of series $\sum_{n=0}^{\infty} u_n$.

Let $\sum_{n=0}^{\infty} u_n$ is a series. Then $\sum_{n=0}^{\infty} u_n$ is said to be summable (C,1) to the sum S if

$$\lim_{n\to\infty} \frac{S_0 + S_1 + \ldots\ldots S_n}{n+1} \to S .$$

Thus if $(EC)_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} C_k^1$

$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k}\sum_{\nu=0}^{k} S_\nu \to S \quad , \quad as \quad n\to\infty$$

where E_n^1 denotes the $(E,1)$ transform of S_n i.e.

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k$$

and S is a finite constant then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(E,1)(C,1)$ means i.e.

$$(EC)_n^1 \to S, \quad (E,1)(C,1) \qquad as \quad n\to\infty .$$

6. CONCLUSION:

Complete work of present research plan will come up in the form of Ph. D. Thesis.

7. WORK ALREADY DONE:

The following work

"ON K^λ - SUMMABILITY OF DERIVED SERIES OF A FOURIER SERIES"

has been completed and accepted for publication in "Bulletin of Calcutta Mathematical Society, Varanasi" which is likely to be published in volume ( ), 2008 of the publication:

8. REFERENCES:

1. Abel, N.H., Undersuchungen über die reine $1+mx+\frac{m(m-1)}{2}x^2+...$, Journal fur die reine and angewandte Mathematik (Crelle), 1(1826), 311-339.
2. Agnew, R.P., The Lotosky method for evaluation of series, Michigan Math. Journal, 4(1957) 105.
3. Alexits, G., Convergence problems of orthogonal series, Pergamon Press, London, 1961.
4. Borwein, D., On product of sequences, Jour. London Math. Soc., 33 (1958), 352 - 357.
5. Burkhurdt, H., Uber den Gebrauch divergenter Reihen in der zeit von 1750-1860, Math Annalen, vol. 70 (1911), 169-206.
6. Cesàro, E., Sur la multiplication des series, Bull. des sciences math, 14 (1890), 114-120.
7. Chandra, P., On the degree of approximation of functions belonging to the Lipschitz class, Nanta Math., 8(1975), 88-89.
8. Fekete, M., On the absolute summability (A) of infinite series, Proceedings of the Edinburg Mathematical Society (2), vol. 3 (1932-33), 132-134.
9. Hardy, G.H., Divergent series, first edition, Oxford University Press, 1949, 70.
10. Hyslop, J.M., On the absolute summability of Fourier series, Proceedings of London Mathematical Society (2), vol. 43 (1937), 475-483.
11. Karamata, J., Theorems sur la Sommability exponentielle etd, autres sommabilities S'y rallachant Mathematica (cluj), 9(1935), 164.
12. Kathal, P.D., A new criteria for Karamata summability of Fourier series, Riv. Math, Univ. Parmo. Italy 10(2) (1969), 33-38.
13. Khare, S.P., Generalised Nörlund summability of Fourier series and its conjugate series, Indian Jour. Pure and Appl. Math., 21(5) (1990), 457-467.
14. Lal, S., On K^{λ} - summability of conjugate Fourier series, Cal. Math. Soc. 89(4) (1997), 327-336.
15. Lal S. and Pratap A., On K^{λ} - summability of Fourier series, Rajasthan Ganit Parishad, 13(1) (1999), 13-18.

16. Lal, S.N., On the Nörlund summability of Fourier series and the behaviour of Fourier coefficients, Indian journal of Mathematics, 13(1971), 177-194.
17. Lorentz, G.G., A contribution to the theory of divergent series. Acta mathematica, 80(1948), 167-190.
18. Lotosky A.V., On a linear transformation of sequences (Russian), Ivanov. Gos. Red. Inst. Uchen. Zap. 4 (1963) 61.
19. Nörlund, N.E., Surune application des functions permutables, Lund. Universite Arsskrift, vol. 16(1919), 1-10.
20. Ojha, M.K., Ph. D. Thesis, BHU, (1982), 120-128.
21. Pati, T., A generalization of a theorem of Iyengar on the harmonic summability of Fourier series, Indian Jour. Math. 3(1961), 85-90.
22. Pandey, B.N., On Nörlund summability of Fourier series and its conjugate series, Indian Jour. Math. Soc., 47 (1983) 87.
23. Prasad, B.N., The absolute summability (A) of Fourier series, Proc. Edinburgh Math. Soc.(2), 2 (1930), 129-134.
24. Prasad, B.N., The summability of a Fourier series, Presidential address, Section of Mathematics and Statistics, 32nd Indian Sci. congress Asso, Nagpur, (1945).
25. Pringsheim, A., Zur Theorie der bestimmten Integrale und der unendlichen Reihen, Math. Annalen. 37(1890), 591-609.
26. Qureshi, K., On degree of approximation of a periodic function f by almost Norlund means, Tamkang Jour. Math 12(1) (1981) 35.
27. Qureshi, K., On degree of approximation of a functions belonging to the Lip α, Indian Jour. Pure App. Math, 13(8) (1982) 898.
28. Qureshi, K. and Neha, H. K., A class of functions and their degree of approximation, Ganita, 41(1) (1990) 37.
29. Rajagopal, C.T., On Nörlund summability of Fourier series, Proc. Camb. Phil. Soc., 59(1963), 47-53.
30. Sahney, B.N. and Goel, D.S., On the degree of continuous functions Ranchi, University Math. Jour., 4 (1973), 50-53.
31. Saxena, A., On uniform harmonic summability of Fourier series and its conjugate series, Proc. Nat. Inst. Sci. India (A), 31(1965), 303-310.

32. Saxena, A., On uniform Nörlund summability of Fourier series, Proc. Nat. Inst. Sci. India Part 32(1965), 502-508.

33. Saxena, A., Nörlund summability of the conjugate series of Fourier series. Abstract, Proc. and 52nd Indian science congress (combined).

34. Singh, T., On the Nörlund summability of Fourier series and its conjugate series, Proc. Natn. Inst. Sci. India, 29(7) (1963), 65-73.

35. Tripathi L.M. and Lal S., On K^{λ}- summability of Fourier series, Journal. Sc. Res. 34 (2) (1984), 69-74.

36. Tripathi, L.M., On Nörlund summability of derived Fourier series, Nat. Inst. Sci. India, 30, (1965), 548-555.

37. Vuĉkoviĉ V., The summability of Fourier series by Karamata methods, Math. Zeitchr. 89 (1965) 192.

38. Woronoi, G.F., Extention of the notion of the limit of the sum of terms of an infinite series, Annals of Mathematics (2), vol. 33(1932), 422-428.

39. Zygmund A., Trigonometric series, 2nd rev. ed., Vol. 1, Cambridge Univ. Press, Cambridge, 1959.

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